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1994 J. Phys. A: Math. Gen. 27 L847

(http://iopscience.iop.org/0305-4470/27/22/001)

J. Phys. A: Math. Gen. 27 (1994) L847–L851. Printed in the UK

LETTER TO THE EDITOR

On kicked systems modulated along the Thue–Morse sequence

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Received 27 July 1994

Abstract. We give necessary conditions for the Fourier transform of the time evolution operator of kicked systems, modulated along the Thue–Morse sequence, to present Dirac's delta functions. The case of two-level systems is analysed in detail and it is found—based on the trace map and numerical investigations—that generically its time evolution has no quasi-periodic component.

Current research on the influence of non-periodic sequences in physical systems has mainly focused on discrete Schrödinger operators, i.e., tight-binding models [1–6]. This is justified by the discovery of quasi-crystals [7]. Since then a number of physicists and mathematicians have been interested in Schrödinger operators with potentials modulated along substitution sequences [1–6, 8], also called *deterministic disorder*. The main tool for characterizing sequences is the Fourier spectrum [8–10], which consists of a countable set in the case of quasi-periodicity. A very interesting substitution sequence is the so-called Thue–Morse sequence, which has a singular continuous Fourier spectrum and so is not quasi-periodic.

Recently some studies on the behaviour of quantum systems driven by non-periodic perturbations have appeared, in particular on systems with time dependence modulated along the Fibonacci [11–13] and Thue–Morse sequences [14, 15], and their dynamical behaviour have been related to the concepts of *quantum integrability* and *quantum chaos*. In the case of N-level systems driven by a quasi-periodic force generated from a Fibonacci sequence (recall that the Fibonacci sequence has a quasi-periodic Fourier spectrum) it has been shown that, generically, the dynamics is not quasi-periodic [11, 12]. This is in contrast to the case of N-level systems driven by time-periodic perturbations in which the time evolution is always quasi-periodic. The case of 2-level systems driven by kicks modulated along the Thue–Morse sequence was considered in [14]; for some parameter values the quantum autocorrelation measure was computed and it splits into a pure point and a pure singular continuous part. However, one has no control of the size of the parameter set for which such interesting results hold. Similar results related to the kicked harmonic oscillator were also found [15].

The purpose of the present letter is to give necessary conditions for the Fourier transform of the evolution operator of kicked systems modulated along the Thue–Morse sequence to present delta functions. Then we study in detail the case of 2-level systems; by using the associated trace map and some numerical calculations we conclude that, in general, the

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Fourier transform of the evolution operator has no delta functions, so that it is not quasi-periodic. As a corollary we see that some results of [14] that appeared particular to the case considered are, in fact, of more general flavour.

We now look at the model. The sequence we consider is $\Gamma = (\nu(k))$, with $k \in \mathbb{N}$ and each $\nu(k)$ takes the value 0 or 1 according to the Thue–Morse substitution rule: $\xi(0) = 10$ and $\xi(0) = 01$. Γ is obtained by writing words from right to left, starting with $\nu(1) = 0$ and iterating ξ at $\nu(1)$; e.g., $\xi^2(\nu(1)) = \xi(\xi(0)) = \xi(10) = \xi(1)\xi(0) = 0110$. If W is a word made of the letters 0 and 1, we shall denote by ${}^tW$ the word obtained by exchanging the letters 0 and 1 in W.

The kicked Hamiltonian we study has the form

$$H(t) = H_0 + P\sum_{k=1}^{\infty} \nu(k)\delta(t-k) \tag{1}$$

where H_0 and P are operators acting in the Hilbert space $\mathcal{H}$. The time-evolution operator $U(t,0)$, generated by Γ, is discontinuous at integer times n such that $\nu(n) = 1$, and by a standard argument one gets

$$U(n,0) \equiv U(n+0, 0+0) = U_{\nu(n)}U_{\nu(n-1)}\dots U_{\nu(1)}$$

where $U_0 = \mathrm{e}^{-\mathrm{i}H_0}$and $U_1 = \mathrm{e}^{-\mathrm{i}H_0}\mathrm{e}^{-\mathrm{i}P}$. It will also be convenient to consider the time-evolution operator $V(t,0)$ generated by ${}^t\Gamma$, so that $V(t,0)$ is discontinuous at integer times n such that $\nu(n) = 0$.

In the case of the kicked system (1) the Fourier transform of the evolution operator $U(n,0)$ can be described by [11] $G(\omega) = \lim_{p\to\infty} G_p(\omega)$, where

$$G_p(\omega) = \sum_{k=1}^{p} \mathrm{e}^{\mathrm{i}\omega k} U_{\nu(k)}U_{\nu(k-1)}\dots U_{\nu(1)}.$$

Set $M_n = U(2^n, 0)$ and ${}^tM_n = V(2^n, 0)$. Two basic consequences of the Thue–Morse substitution rule are $M_{n+1} = {}^tM_n M_n$ and ${}^tM_{n+1} = M_n {}^tM_n$ [14]. Making use of these relations it is found that

$$G_{2^{n+1}}(\omega) = G_{2^n}(\omega) + {}^tG_{2^n}(\omega)\mathrm{e}^{\mathrm{i}\omega 2^n} M_n \tag{2}$$

$${}^tG_{2^{n+1}}(\omega) = {}^tG_{2^n}(\omega) + G_{2^n}(\omega)\mathrm{e}^{\mathrm{i}\omega 2^n}\, {}^tM_n \tag{3}$$

where

$${}^tG_p(\omega) = \sum_{k=1}^{p} \mathrm{e}^{\mathrm{i}\omega k} V(k,0)$$

is obtained from $G_p(\omega)$ by exchanging the letters 0 and 1. $G(\omega)$ has δ-singularities for the values of ω such that $\phi_n(\omega) \equiv G_{2^n}(\omega)/2^n$ has a non-zero limit $\phi(\omega)$ as $n \to \infty$ [11, 16]. Taking the limit $n \to \infty$ in (2) and (3) divided by 2^{n+1} one obtains, respectively,

$$\phi(\omega) = {}^t\phi(\omega)L(\omega) \qquad \text{and} \qquad {}^t\phi(\omega) = \phi(\omega)\,{}^tL(\omega) \tag{4}$$

where we have introduced $L(\omega) = \lim_{n\to\infty} \mathrm{e}^{\mathrm{i}\omega 2^n} M_n$, ${}^tL(\omega) = \lim_{n\to\infty} \mathrm{e}^{\mathrm{i}\omega 2^n}\,{}^tM_n$, and the obvious notation ${}^t\phi(\omega)$.

For simplicity we have considered only strong limits of operators and excluded particular cases in which, for instance, zero is an eigenvalue of $\phi(\omega)$, but $\phi(\omega)$ is not the zero operator on $\mathcal{H}$; another instance we have excluded from our discussion is the case where U_1 and U_0 do not commute but have a common eigenvector. Such cases reduce to the analysis on particular sub-spaces of $\mathcal{H}$ and ought to be handled separately; they are left to the interested reader.

From relation (4) one sees that $\phi(\omega)$ is non-zero if and only if ${}^t\phi(\omega)$ is non-zero and, in this case, $L(\omega)$ and ${}^tL(\omega)$ are well defined. Notice that $L(\omega)$ and ${}^tL(\omega)$ are never zero since they are (strong) limits of unitary operators.

Suppose that $\phi(\omega)$ is non-zero. Then from relation (4) one finds that $L(\omega)\,{}^tL(\omega)\,{}^tL(\omega)L(\omega) = I$ (identity operator). If we write out this relation we obtain

$$I - {}^tL(\omega)L(\omega) = \lim_{n\to\infty}(\mathrm{e}^{\mathrm{i}\omega 2^{n+1}}\,{}^tM_n M_n) = \lim_{n\to\infty}\mathrm{e}^{\mathrm{i}\omega 2^{n+1}} M_{n+1} = L(\omega).$$

In a similar way we find ${}^tL(\omega) = I$. Therefore, a necessary condition for a δ function concentrated at ω in the Fourier transform G is

$$L(\omega) = {}^tL(\omega) = I. \tag{5}$$

Notice that from relation (5) it follows that the presence of δ functions in G implies that $0 \neq \phi(\omega) = {}^t\phi(\omega)$. We also have

$$0 = {}^tL(\omega) - L(\omega) = \lim_{n\to\infty}\mathrm{e}^{\mathrm{i}\omega 2^n}({}^tM_n - M_n)$$

so that

$$\lim_{n\to\infty}({}^tM_n - M_n) = 0.$$

This last condition does not depend on ω and seems to be non-generic. Next we discuss the case of 2-level systems.

Now we check that for 2-level systems subject to kicks generated by the Thue–Morse substitution sequence condition (5) is not generic, so that its time evolution has, in general, no quasi-periodic component. The system is given by (1) with $H_0 = \varepsilon\sigma_z$, $P = \lambda\sigma_x$, $\varepsilon, \lambda \in \mathbb{R}$ and $\mathcal{H} = \mathbb{C}^2 - \sigma_x$ and σ_z are the standard Pauli matrices.

The main ingredient here is the trace map, a fundamental tool for the study of tight-binding Schrödinger equation with disorder induced from substitution sequences [1, 2, 5, 17]. Let $x_n = \mathrm{Trace}(M_n)/2$; it was shown by Combescure [14] that for $n \geqslant 2$

$$x_{n+1} = 1 + 4x_{n-1}^2(x_n - 1) \tag{6}$$

and the initial conditions have the form

$$x_1 = \cos(2\varepsilon)\cos\lambda \qquad x_2 = 1 - 2\cos^2\varepsilon(1 + \cos^2\lambda(3 - 4\cos^2\varepsilon)). \tag{7}$$

Since M_n are unitary 2×2 matrices it follows that $x_n \in [-1, 1]$.

It was proven [14] that if $x_n = 1$ for some n, then for any non-zero initial state the quantum autocorrelation measure of this 2-level system is the sum of a pure point and of a singular continuous measure. An open problem is to show if the condition $x_n = 1$ holds generically (in some particular sense, e.g., second category set or positive Lebesgue

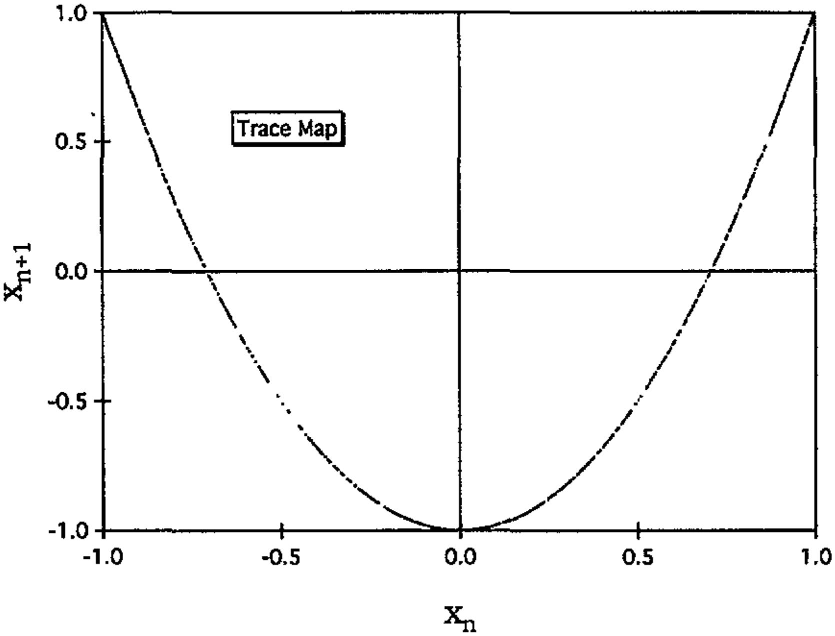

Figure 1. 1000 pairs of consecutive iterates from the trace map (6) after the transient has been ruled out. The initial condition is given by (7) with $\varepsilon = 5.8933$ and $\lambda = 3.669$. These values of ε and λ were picked up at random.

measure) in the parameter space. We claim that the condition $x_n = 1$ is not generic; our argument is as follows.

Set $\rho_n(\omega) = e^{i\omega 2^n} x_n$. If the Fourier transform G has a δ function at ω it follows from (5) that $\rho_n(\omega) \to 1$ for $n \to \infty$. From the trace map (6) we get the recurrence relation for $\rho_n(\omega)$

$$\rho_{n+1}(\omega) - e^{i\omega 2^{n+1}} = 4\rho_{n-1}^2(\omega)(\rho_n(\omega) - e^{i\omega 2^n}).$$

The condition $\rho_n(\omega) \to 1$ implies $\lim_{n\to\infty} e^{i\omega 2^n} = 1$, so that the ω values in the interval $[0, 2\pi]$ are restricted to $\omega_{jk} = 2\pi j/2^k$, with j, k positive integers, $j \leqslant 2^k$. In [14, 15] the δ functions in the correlation measures were concentrated at frequencies of the form ω_{jk}; we have just shown that for 2-level systems these values of frequencies cover all possibilities.

For n large enough $\rho_n(\omega_{jk}) = x_n$, and a necessary condition for the presence of a δ function in the Fourier transform is

$$\lim_{n\to\infty} x_n = 1. \tag{8}$$

This condition is equivalent to $\lim_{n\to\infty} M_n = \lim_{n\to\infty} {}^t M_n = I$ [14]. Note that Combescure's results hold under the hypothesis $M_n = I$ for some n, and that the substitution rule implies that $M_s = {}^t M_s = I$ for any $s > n$.

We have therefore arrived at the problem of characterizing the base of attraction, in the parameter space, of the fixed point $\zeta = 1$ of the trace map (6) (with initial conditions (7)). Notice that the dynamics

$$x_{n+1} = 2x_n^2 - 1 \tag{9}$$

is an 'invariant' of the trace map, i.e., if (9) is satisfied for the pair (x_n, x_{n+1}) then (6) implies that (x_{n+1}, x_{n+2}) also satisfies (9). The map (9) is a well known unimodal map on the interval $[-1, 1]$ that has a unique invariant measure μ absolutely continuous with respect to Lebesgue measure [18]. Furthermore, μ is an attractor [19] for (9) in the sence that the ergodic averages with initial condition concentrated on $y \in [-1, 1]$ converge to μ for y in a set of full Lebesgue measure—in fact μ is the unique attractor for (9). We have then a hint that μ could also be an attractor for the trace map; we have checked this point numerically and have found that this is actually the case. In figure 1 we show the last 1000 iterates of the trace map from a total of 6000 with initial conditions (7) with $\varepsilon = 5.8933$ and $\lambda = 3.669$ (The numerical precision has to be controlled in order to avoid overflow of the trace map iterates.). After a somewhat short transient the iterates of (6) converge to the corresponding iterates of (9). This has occurred for all pairs of parameters (ε, λ) we have checked.

From the above results we conclude that μ is an attractor for the trace map. Therefore, the set of points in the parameter space such that x_n converges to the fixed point ζ as $n \to \infty$ is not generic, as well as the presence of a quasi-periodic component in the time evolution of the two level system corresponding to (1), Although our last argument is supported only by numerical calculations, these are simple enough to be considered very strong.

This work was supported in part by Fundação de Amparo à Pesquisa do Estado de São Paulo FAPESP (Brazil) under contract 93/2853-5, that provided me with computing facilities.

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